AIAA JOURNAL
Vol. 30, No. 11, November 1992

Method of Simulating Unsteady Turbomachinery Flows with Multiple Perturbations

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Currently time-marching methods for unsteady turbomachinery flow calculations suffer from the inability to deal with multiple perturbations in a single-passage solution. A new periodic boundary condition method has been proposed, in which one can perform a single-passage solution with multiple perturbations. At the periodic boundaries, all of the perturbations are identified by their own phase-shifted periodicities and approximated by Fourier series. A combined parameter is chosen for the Fourier transform so that the Fourier coefficients can be efficiently updated. Corrections on both the variables at a current time level and on the Fourier coefficients are carried out repeatedly until a periodic state is reached. Numerical examples for oscillating cascade flows subject to inlet/outlet perturbations have demonstrated the validity of the present method and its ability to deal with situations of practical interest.

Nomenclature

A_m = amplitude
C = length of blade chord
C_{p_1} = unsteady pressure coefficient
f = frequency
K = reduced frequency, $\omega c/u_1$
M = Mach number
N_{fou} = highest order of the Fourier components
N_p = number of time steps in one period
N_{pt} = number of perturbations
P = pressure
P^0 = steady inlet stagnation pressure
R = gas constant
r = radial coordinate
S = entropy
T = temperature
t = time
U = primitive flow variable
u_r = circumferential traveling speed of perturbation
u_1 = inlet velocity
x = axial coordinate
y = circumferential coordinate
Δt = time-step increment
ΔY_p = pitch length
σ = interblade phase angle
ω = angular frequency

Subscripts

b = bending vibration
i = index of perturbation
n = index of the order of the Fourier harmonic
t = torsion vibration
0 = time-averaged value
2 = outlet boundary

Superscripts

L = lower periodic boundary
U = upper periodic boundary
0 = stagnation parameter

Received Feb. 6, 1992; revision received May 19, 1992; accepted for publication May 26, 1992.

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Introduction

PREDICTIONS of unsteady turbomachinery flows due either to rotor/stator interaction or blade vibration have attracted considerable attention. Corresponding solution methods can be developed using either time-linearized models in which unsteadiness is assumed to be a small perturbation to steady flow,[1-3] or nonlinear time-marching techniques.[4-6,8] It is well recognized that nonlinear time-marching calculations, although making fewer assumptions, are much more expensive than time-linearized solutions.

A typical problem for time-marching calculations for unsteady turbomachinery flows is how to implement a periodic boundary condition, if the computation is conducted in a single blade passage, as shown in Fig. 1. For blade row interaction problems, if the rotor and stator rows have different blade numbers as in all practical situations, the upper periodic boundary (a-b or e-f) will sense what the lower boundary (c-d or g-h) senses with a constant phase lead or lag in time, i.e., there is a phase-shifted periodicity. Similarly, for blade flutter problems adjacent blades usually vibrate with a constant phase angle difference [interblade phase angle (IBPA)]; therefore the phase-shifted periodic condition must also be applied at the upper and lower periodic boundaries.

Several methods have been developed to deal with the phase-shifted periodicity. The first method, proposed by Erdos et al.,[7] is known as "direct store." In this method, flow variables at the periodic boundaries are stored for one period of time. Then the stored parameters and current solutions correct each other according to the phase-shifted periodicity. A main disadvantage of the direct store method is that a large amount of computer storage is usually required. Giles[5] proposed a space-time transformation method to implement the phase-shifted periodic condition. In his method the time plane in the computational domain is inclined along the blade pitchwise direction according to a given interblade phase angle. The phase-shifted periodic condition can then be directly applied

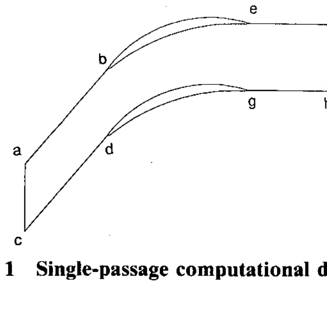


Fig. 1 Single-passage computational domain.

by equating flow variables at the periodic boundaries. Therefore no extra storage is required, and the convergence rate is faster because the solution procedure is less influenced by the initial guess, compared with the direct store method. However, the time-inclination angles (and therefore allowed interblade phase angles) are restricted by the characteristics of the governing equations. The present author[6] proposed a method called "shape correction." Flow variables at the periodic boundaries are transformed into the Fourier components. Instead of directly storing flow parameters as in the direct store method, only the Fourier components are stored, so computer storage is greatly reduced. The stored temporal "shape" of flow variables is then used to correct the current solution at the periodic boundaries.

It must be pointed out that all of the aforementioned periodic boundary condition methods, although different in various aspects, have one common feature; that is, they can only be used to deal with unsteady flow problems with a single perturbation, e.g., either due to another relatively moving blade row or due to blades oscillating in a single mode in an isolated row. In other words, none of the methods are capable of performing a single-passage solution for unsteady flows either due to more than one relatively moving blade row with nonequal blade number or due to oscillating blades in a single blade row with arbitrary inlet and outlet periodic disturbances. In practice, several unsteady perturbations with their own phase-shifted periodicities coexist, and their interactions between each other and with the steady flow are of great interest. To the author's knowledge, this kind of general unsteady turbomachinery flow with multiple perturbations can so far only be dealt with by performing the multiple-blade-passage (most probably the whole annulus) calculations, which are much beyond current computing ability. The time-linearized methods, on the other hand, can handle the multiple perturbations simply by superposition. Thus the difference in solution efficiency between a nonlinear time-marching solver and a time-linearized solver would be greatly increased for multiple perturbation problems. This difficulty of dealing with multiple perturbations in a single-passage solution can be regarded as a very serious limitation on further applications of nonlinear time-marching methods to unsteady turbomachinery flows.

The present paper describes a new generalized shape correction method for the phase-shifted periodic boundary condition, which allows a single-blade-passage solution for unsteady turbomachinery flows with multiple perturbations. The description of the methodology will be followed by two test cases with essentials representative of general situations of interest.

Methodology

Assumptions and Basic Formulations

It is assumed that all of the unsteady perturbations to be dealt with satisfy a temporal and circumferentially spatial periodicity, characterized by a spatial wavelength (and its higher order harmonics) and a constant traveling velocity along the circumferential direction. It is also assumed that any flow variable at periodic boundaries can be expressed in the following form:

$$U(x, y, r, t) = U_0(x, y, r) + \sum_{i=1}^{N_{pt}} U_i(x, y, r, t) \tag{1}$$

where U_0 is the time-averaged part and U_i is the unsteady part with the temporal and spatial periodicity of the perturbation i.

Similar to the shape correction method for single perturbation problems,[6] each unsteady part is approximated by a Fourier series in time:

$$U_i(x, y, r, t) = \sum_{n=1}^{N_{fou}} [A_{ni}(x, y, r)\sin(n\omega_i t + \phi) + B_{ni}(x, y, r)\cos(n\omega_i t + \phi)] \tag{2}$$

Then,

$$U = U_0 + \sum_{i=1}^{N_{pt}} \sum_{n=1}^{N_{fou}} [A_{ni}\sin(n\omega_i t + \phi) + B_{ni}\cos(n\omega_i t + \phi)] \tag{3}$$

Care should be taken when choosing the number of perturbations N_{pt}. Let us regard the perturbations with the phase-shifted periodicities defined by moving boundaries (e.g., rotating blade rows or oscillating blades with given interblade phase angle and frequency) as fundamental perturbations. For situations where nonlinear interactions between fundamental perturbations are not very strong, N_{pt} should be the same as the number of fundamental perturbations concerned. Then interactions between the different perturbations and the steady flow manifest themselves in amplitudes and phases of corresponding parts. Whereas if the nonlinear interaction is very strong, unsteadiness with frequencies of $(\omega_i + \omega_j)$, $(\omega_i - \omega_j)$, $(2\omega_i + \omega_j)$, $(2\omega_i - \omega_j)$, . . . , $(i \neq j)$, might be induced, in the form of products of fundamental perturbations. For instance, if we have two perturbations in the form of

$$f = f_0 + f_1 \sin \omega_1 t + f_2 \sin \omega_2 t$$

then the quadratic (nonlinear) term will be

$$f^2 = f_0^2 + ½(f_1^2 + f_2^2) + 2f_0 f_1 \sin \omega_1 t + 2f_0 f_2 \sin \omega_2 t$$
$$- ½(f_1^2 \text{ cons } 2\omega_1 t + f_2^2 \cos 2\omega_2 t) + f_1 f_2 \cos(\omega_1 - \omega_2)t$$
$$- f_1 f_2 \cos(\omega_1 + \omega_2)t$$

The last two terms are the induced perturbations in the form of product between the two fundamental perturbations. The induced perturbations should be included in Eq. (3) if one wants a complete description of corresponding unsteady terms. In practice, however, only low harmonics tend to have strong interactions. Therefore inclusion of a few induced perturbations with low frequencies should be adequate for engineering purposes.

Based on the assumptions about the temporal and spatial periodicity, we have

$$U = U_0 + \sum_{i=1}^{N_{pt}} \sum_{n=1}^{N_{fou}} \left[A_{ni} \sin\left(n\omega_i t - \omega_i \frac{\Delta y}{u_{ri}} \right) + B_{ni} \cos\left(n\omega_i t - \omega_i \frac{\Delta y}{u_{ri}} \right) \right] \tag{4}$$

where u_{ri} is the constant traveling speed of the ith circumferential perturbation mode and Δy is the circumferential distance from a reference point.

Accordingly, for any pair of mesh points at upper U and lower L periodic boundaries of a computational domain for a single blade passage, we have

$$U^L(x, r, t) = U_0(x, r) + \sum_{i=1}^{N_{pt}} \sum_{n=1}^{N_{fou}} [A_{ni}\sin(n\omega_i t) + B_{ni}\cos(n\omega_i t)] \tag{5}$$

$$U^U(x, r, t) = U_0(x, r) + \sum_{i=1}^{N_{pt}} \sum_{n=1}^{N_{fou}} [A_{ni}\sin(n\omega_i t + \sigma_i) + B_{ni}\cos(n\omega_i t + \sigma_i)] \tag{6}$$

where σ_i is the interblade phase angle for the ith perturbation.

Implementation

Following the principal idea of the shape correction method for single perturbation,[6] a time-marching solution procedure can be started with an assumed temporal shape at each periodic boundary point. Then we can use the stored shape and the current solution to correct each other according to the phase-shifted periodicity until the solution converges to a periodic state.

For cases with a single perturbation, corrections to the stored shape (i.e., the Fourier coefficients A_n and B_n) can be easily conducted by a straightforward timewise integration for parameters from the current solution, e.g.,

$$A_n = \frac{\omega}{\pi} \sum_{1}^{N_p} U \sin(n\omega t)\, \Delta t \tag{7a}$$

$$B_n = \frac{\omega}{\pi} \sum_{1}^{N_p} U \cos(n\omega t)\, \Delta t \tag{7b}$$

Therefore, A_n and B_n can be updated once a period. However, for multiple perturbations, equivalent formulations to Eqs. (7) for each perturbation can only be used if the frequencies of all perturbations concerned can beat each other after a certain number of periods. For example, if $f_1 = 2$ Hz and $f_2 = 3$ Hz, then the beating frequency is 1 Hz, so

$$A_{n1} = \frac{\omega_1}{2\pi} \sum_{1}^{2N_{P1}} U \sin(n\omega_1 t)\, \Delta t \tag{8a}$$

$$A_{n2} = \frac{\omega_2}{3\pi} \sum_{1}^{3N_{P2}} U \sin(n\omega_2 t)\, \Delta t \tag{8b}$$

Thus A_{n1} and B_{n1} have to be updated once every two periods of the first perturbation, and three periods are needed for updating A_{n2} and B_{n2}. In practice, the beating period can be much longer than the periods of perturbations concerned. Therefore the time-marching solution can be greatly slowed down.

What we really need is that the Fourier coefficients for each perturbation can be updated and corrected in as short time as possible. We know that the following formulations should hold for a converged solution at the same order of approximation as Eqs. (7) for a single perturbation case

$$A_{ni} = \frac{\omega_i}{\pi} \sum_{1}^{N_{pi}} (U - R_i)\sin(n\omega_i t)\, \Delta t \tag{9a}$$

$$B_{ni} = \frac{\omega_i}{\pi} \sum_{1}^{N_{pi}} (U - R_i)\cos(n\omega_i t)\, \Delta t \tag{9b}$$

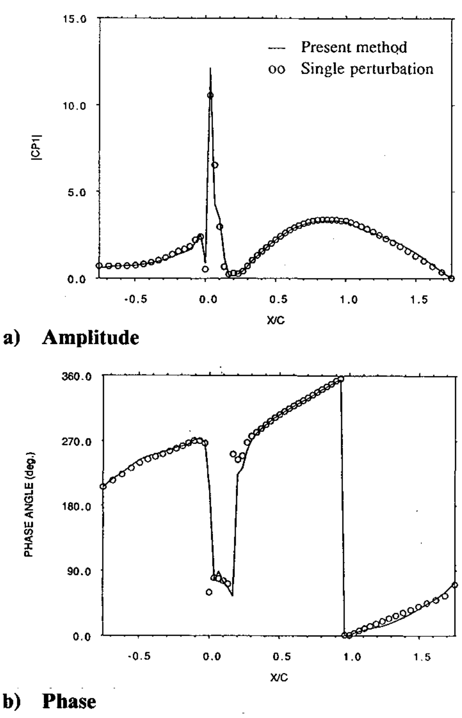


Fig. 2 Unsteady pressure along lower periodic boundary (flat plate, pitchwise bending mode).

where

$$R_i = \sum_{j \neq i}^{N_{\rm pt}} \sum_{n=1}^{N_{\rm fou}} [A_{nj} \sin(n\omega_j t) + B_{nj} \cos(n\omega_j t)] \tag{10}$$

The idea is that, if we can approximate R_i, we would be able to update and correct A_{ni} and B_{ni} once per period of the corresponding perturbation. An apparent choice of the approximation for R_i would be the stored shape. Therefore we will not only use the stored shape to correct the current solution but also use some part of it together with the current solution to form a parameter to work out new values of the Fourier coefficients. In this way we induce an additional source of error, which may affect the convergence rate but not the final converged results.

The step-by-step implementation is as follows:

Step 0: Make an initial guess for U_0 and A_{ni}, B_{ni}. Then start the unsteady solution.

Step 1: At time step n, update the primitive variable U at the periodic boundaries from the current solution.

Step 2: For each perturbation, use U and previous Fourier coefficients for other perturbations to conduct a timewise integration [Eq. (9)] for new coefficients.

Step 3: Correct U using the previous Fourier coefficients [Eqs. (5) and (6)].

For each perturbation, new values of the Fourier coefficients are obtained once per period and then used to correct old ones. Acceleration of updating the Fourier coefficients can be made by carrying out several timewise integrations starting at different moments in one period of the corresponding perturbation.

Numerical Examples

Two test cases for two-dimensional unsteady cascade flows will be presented. The calculations were carried out using an inviscid Euler solver[6] in a single-blade passage with the periodic boundary condition being implemented by the present method. In addition to unsteadiness due to blade vibration, inlet stagnation pressure and outlet static pressure perturbations were also included. The far-field pressure perturbations were specified in the form of

$$P_b = P_{b0}\left[1 + A_{mp} \sin\left(\omega_p t - \omega_p \frac{\Delta y}{u_{rp}}\right)\right] \tag{11}$$

with the corresponding IBPA being

$$\sigma_p = -\frac{\omega_p \Delta Y_p}{u_{rp}}$$

where P_b is either the inlet stagnation pressure P^0 or the outlet static pressure P_2. The far-field boundary conditions were implemented in the conventional steady flow method; i.e., at inlet, specify P^0, T^0, and flow angle for subsonic flow or circumferential velocity for supersonic flow; and at outlet, specify static pressure P_2. These boundary condition treatments do reflect outgoing waves. But they were chosen here simply for convenience of defining the perturbations at the boundaries. In the previous Euler solver for single perturbation problems,[6] which had been validated against some well-documented linear theories and experiments, a one-dimensional nonreflecting condition and an approximate two-dimensional nonreflecting condition have been adopted for the far-field boundaries. It was found that the accuracy of the near-field solution (e.g., surface pressure) would only be considerably affected by the far-field boundary condition treatment under an acoustically cut-on condition. Here, for consistency, the same reflecting far-field boundary conditions as just mentioned were adopted for both the single-perturbation and the multiple-perturbation solutions.

It should be mentioned that, for unsteady flows with multiple perturbations, the convergence of the solution is indicated by the periodicity with the beating frequency. To obtain a periodic solution, twice as many time steps as in a single perturbation situation are usually needed.

Case 1: Subsonic Flate Plate Cascade

The first case is of flat plate cascade at a subsonic flow condition. The cascade geometry and steady flow conditions are

Chord = 0.0762 m
Chord/pitch ratio = 1.3
Stagger angle = 45 deg
Inlet Mach number = 0.65
Incidence = 0 deg

The cascade is oscillating in a pitchwise bending mode, subject to an inlet stagnation pressure perturbation and an outlet static pressure perturbation, defined in Eq. (11). The parameters for these three modes are the following:

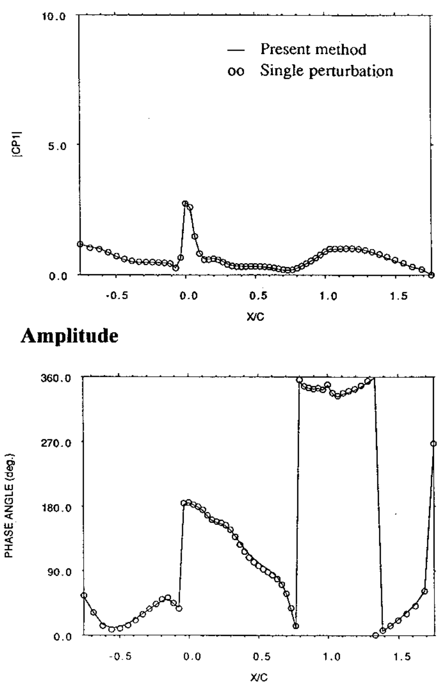
Mode 1 (pitchwise bending vibration):
Amplitude $= A_{mb} = 0.01\ C$
Frequency $= f_b = 666.667$ Hz ($K = 1.52$)
IBPA $= \sigma_b = 50$ deg

Mode 2 (inlet upward-traveling stagnation pressure wave):
Amplitude $= A_{mp0} = 0.03$
Frequency $= f_{p0} = 1333$ Hz ($K = 3.04$)
IBPA $= \sigma_{p0} = -300$ deg

Mode 3 (outlet upward-traveling static pressure wave):
Amplitude $= A_{mp2} = 0.03$
Frequency $= f_{p2} = 1000$ Hz ($K = 2.28$)
IBPA $= \sigma_{p2} = -270$ deg

For this case nonlinear effects are expected to be negligible, so only the first-order harmonic components were taken in the Fourier series for the periodic boundary condition. The combined modes have 333.3 Hz as the beating frequency. Once a periodic state with the beating frequency was reached, the unsteady pressure was decoupled into the three modes by Fourier transforms. These results were then compared with those by the single perturbation shape correction method[6] at each perturbation condition. To more clearly check the validity of the periodic boundary condition treatment, the comparisons of the results were made along the lower periodic boundary of the computational domain from the inlet to the outlet (i.e., the line c-d-g-h as shown in Fig. 1, where d-g is on the blade upper surface).

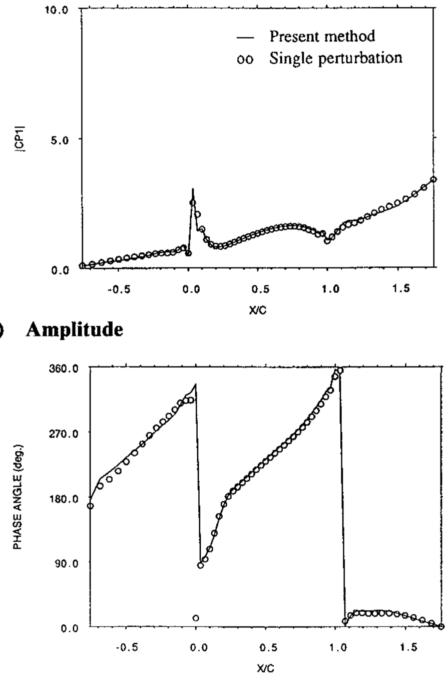
The decoupled unsteady pressure results for mode 1 (blade vibration) are compared with the single perturbation results in Fig. 2 (note that the region with $0 < x/c < 1$ is on the blade surface). The results for mode 2 (inlet P^0 perturbation) are given in Fig. 3. The results for mode 3 (outlet P_2 perturbation) are shown in Fig. 4. The agreement between the present decoupled results and those directly from the single perturbation solution is very good for all of the three modes.



a) Amplitude

b) Phase

Fig. 3 Unsteady pressure along lower periodic boundary (flat plate, inlet P^0 mode).



a) Amplitude

b) Phase

Fig. 4 Unsteady pressure along lower periodic boundary (flat plate, outlet P_2 mode).

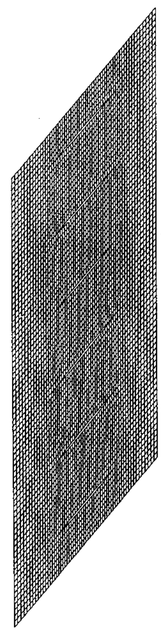
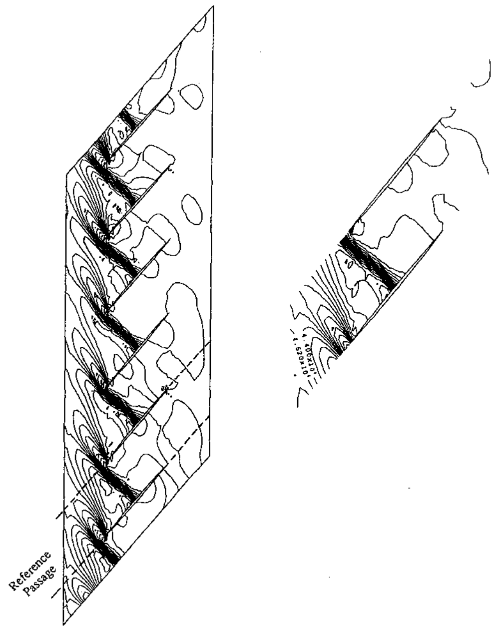


Fig. 5 Computational mesh of the six-passage solution.

Case 2: Transonic Biconvex Cascade

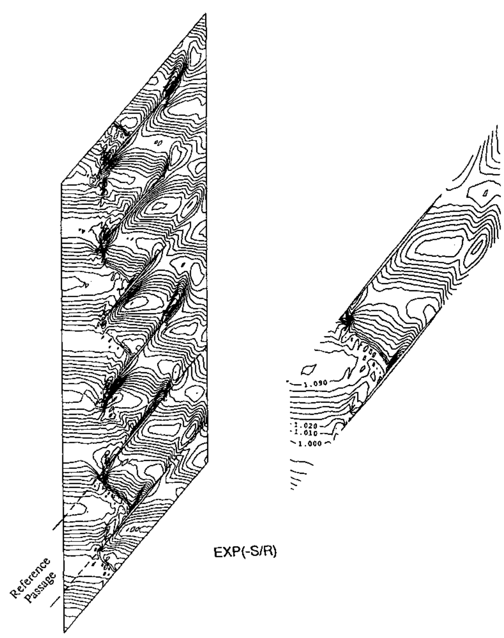
To further check the present method, an unsteady flow case with more practical conditions was calculated. The cascade geometry and steady flow conditions are similar to those for a typical low-pressure compressor or fan tip section:

Inlet Mach number = 1.15
Back pressure (P_2/P^0) = 0.67
Stagger angle = 50 deg
Chord = 0.1 m
Chord/pitch ratio = 1.3



a) Multipassage b) Present solution

Fig. 6 Static pressure contour ($\omega t = 0$ deg, interval = 2200).



a) Multipassge b) Present solution

Fig. 7 Entropy contour ($\omega t = 0$ deg, interval = 0. 01).

The cascade was subject to two specified perturbations: blade vibration and inlet stagnation perturbation. The parameters for these two modes were chosen to be within the range of practical interest. They are the following:

Mode 1 (blade vibration in torsion around midchord):

Amplitude	$= A_{mt} = 1$ deg
Frequency	$= f_t = 400$ Hz ($K = 0.71$)
IBPA	$= \sigma_t = 60$ deg

Mode 2 (inlet stagnation pressure wave):

Amplitude	$= A_{mp0} = 0.05$
Frequency	$= f_{p0} = 2400$ Hz ($K = 4.26$)
IBPA	$= \sigma_{p0} = -240$ deg
Traveling speed	$= 277$ m/s

To have a more direct check on the present periodic boundary condition method, this case was first calculated by a multiple-passage solution. The computational domain covers six passages (Fig. 5). Clearly the unsteady flow has a spatial periodicity for six blade passages, so that the same direct periodic condition as in steady blade-to-blade flow computations can be applied by equating variables at the upper and lower boundaries for the multipassage solution. Hence the multipassage solution has the least approximation for the periodic boundary condition and can be used as a baseline to validate the present single-passage solution.

Figure 6a shows static pressure contours of the multipassage solution at the instant $\omega t = 0$ deg, after a periodic state with the beating frequency ($f = 400$ Hz) was reached. The corresponding entropy contour at the same instant in term of $\exp(-S/R)$ is shown in Fig. 7a. We can observe from these contours that a strong normal shock standing on the suction surface intersects the periodic boundary upstream of the leading edge of the adjacent blade. Thus this is a challenging test case for periodic boundary condition treatment.

The present single-passage solution was carried out with the same mesh density as that for the multipassage solution. To cope with nonharmonic unsteadiness due to shock motion, Fourier components up to the fifth order were included to describe temporal variations of flow variables at the periodic boundaries for the two modes. When the single-passage solution converged to a periodic state with the beating frequency, we also plotted the static pressure (Fig. 6b) and entropy (Fig. 7b) contours at $\omega t = 0$ deg. Figure 6b can be compared with the static pressure contour in the region marked "Reference Passage" in Fig. 6a, and Fig. 7b can be compared with the entropy contour in the region marked "Reference Passage" in Fig. 7a. These contours from both the multipassage and single-passage solution are in good agreement. It should also be pointed out that the single-passage results (Figs. 6b and7b) show almost no sign of reflection of pressure and entropy waves from the periodic boundaries.

Similar to case 1, unsteady pressure results were taken along the lower periodic boundary mesh line and decoupled into the two modes by a Fourier transform. The amplitude and phase distributions of the first harmonic of unsteady pressure coefficient for mode 1 (blade torsion vibration) from both the single-passage and multipassage solutions are compared as shown in Fig. 8. The comparisons of the first harmonic unsteady pressure for mode 2 (inlet stagnation pressure wave) are

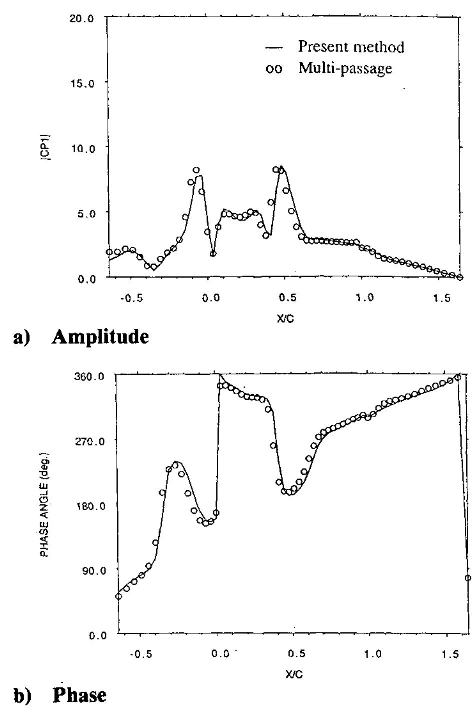


Fig. 8 Unsteady pressure along lower periodic boundary (biconvex cascade, torsion vibration mode).

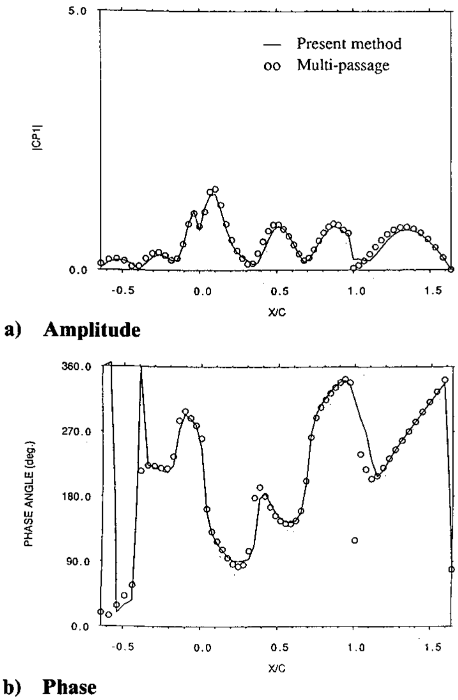


Fig. 9 Unsteady pressure along lower periodic boundary (biconvex cascade, inlet P^0 mode).

shown in Fig. 9. The agreement between the results using the two different periodic boundary condition methods can be regarded as quite satisfactory. These comparisons not only demonstrate the validity of the present methodology but also indicate the feasibility of the method for applications to practical flow conditions.

Concluding Remarks

Previous time-marching methods for unsteady turbomachinery flow calculations suffer from being unable to deal with multiple perturbations in a single-passage solution.

In this paper, a new periodic boundary condition method has been proposed that permits a single-passage solution with multiple perturbations. At the periodic boundaries of a single-passage computational domain, all of the perturbations are identified by their own phase-shifted periodicities and approximated by Fourier series. Following the shape correction methodology, the variables of the current solution are corrected using the previous Fourier coefficients. The Fourier coefficients are efficiently updated by taking a Fourier transform for a combination of the current solution and the previous Fourier coefficients. This correction procedure is repeated until a periodic state is reached.

The present method has been implemented in an Euler solver and has been validated for a subsonic oscillating flat plate cascade flow subject to inlet stagnation pressure and outlet static pressure waves and an oscillating biconvex cascade with a normal shock standing off the leading edge subject to an inlet stagnation pressure perturbation. The results for the flat plate cascade are compared with a single perturbation solution. The results for the transonic cascade are compared with a multiple-passage solution. All of the comparisons are very favorable. The ability to deal with practical situations has also been demonstrated.

Acknowledgment

The present work has been supported by Rolls-Royce. The author wishes to thank Peter Stow, Arj Suddhoo, and Alex Cargill of Rolls-Royce for their technical support. The author is very grateful to John Denton for his encouragement and guidance.

References

[1]Verdon, J. M., and Caspar, J. R., "Development of a Linear Unsteady Aerodynamics for Finite-Deflection Cascades," *AIAA Journal*, Vol. 20, No. 9, 1982, pp. 1259–1267.

[2]Whitehead, D. S., "The Calculations of Steady and Unsteady Transonic Flow in Cascades," Cambridge Univ., Engineering Dept., Rept. CUED/A-Turbo/TR 118, Cambridge, England, UK, 1982.

[3]Hall, K. C., and Crawley, E. F., "Calculation of Unsteady Flows in Turbomachinery Using the Linearized Euler Equations," *AIAA Journal*, Vol. 27, No. 6, 1989, pp. 777–787.

[4]Hodson, H. P., "An Inviscid Blade-to-Blade Prediction of a Wake-Generated Unsteady Flow," American Society of Mechanical Engineers, ASME Paper 84-GT-43, New York, June 1984.

[5]Giles, M. B., "Calculation of Unsteady Wake Rotor Interaction," *Journal of Propulsion and Power,* Vol. 4, No. 4, 1988, pp. 356–362.

[6]He, L., "An Euler Solution for Unsteady Flows around Oscillating Blades," *ASME Journal of Turbomachinery,* Vol. 112, No. 4, 1990, pp. 714–722.

[7]Erdos, J. I., Alzncr, E., and McNally, W., "Numerical Solution of Periodic Transonic Flow Through a Fan Stage," *AIAA Journal,* Vol. 15, No. 11, 1977, pp. 1559–1568.

[8]He, L., and Denton, J. D., "Inviscid-Viscous Coupled Solution for Unsteady Flows Through Oscillating Blades, Part 1, Description of the Method," American Society of Mechanical Engineers, ASME Paper 91-GT-125, New York, June 1991; also, *ASME Journal of Turbomachinery* (to be published).